

IRREDUCIBILITY AND UNIQUENESS OF STATIONARY DISTRIBUTION

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ABSTRACT. In this paper, we shall prove that the irreducibility in the sense of fine topology implies the uniqueness of invariant probability measures. It is also proven that this irreducibility is strictly weaker than the strong Feller property plus irreducibility in the sense of original topology, which is the usual uniqueness condition.

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1. INTRODUCTION

The existence and uniqueness of invariant measures have been one of the most important problems in theory of Markov processes. Let (P_t) be a transition semigroup on a measurable space (E, \mathcal{E}) . A σ -finite measure μ is invariant if $\mu P_t = \mu$ for any $t > 0$, where

$$\mu P_t(A) := \int \mu(dx) P_t(x, A).$$

An invariant probability measure is also called an invariant distribution or stationary distribution. The existence of an invariant distribution usually means the positive recurrence and the uniqueness means ergodicity.

It is well-known (see e.g. [2], [3], [5]) and also very useful that for a nice Markov process on a nice topological space, the strong Feller property (P_t takes bounded measurable function to continuous function) together with the irreducibility (any point can reach any open set) implies the uniqueness of invariant distribution. Usually the irreducibility is intuitive and not very hard to check. However it seems that the strong Feller is really strong in many cases especially in degenerate cases. Besides, two conditions involve the topology much more than the invariant measure itself does, and therefore are not so essential.

In this paper we are going to introduce another irreducibility which is more natural in some sense. For example it depends on the topology induced by the process itself. We shall prove that the irreducibility implies the uniqueness of invariant distribution

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and also prove that it is really weaker than the strong Feller property plus the irreducibility (in the sense of original topology). We also give some characterizations which are easy to check.

2. MAIN RESULTS

Let

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$$

be a right Markov process on (E, \mathcal{E}) (say, Polish), with transition semigroup (P_t) and resolvent (or potential operator) $(U^\alpha : \alpha \geq 0)$. For any nearly Borel set B , T_B always denotes the hitting time of B .

Definition 2.1. *X is called irreducible if $\mathbf{P}^x(T_G < \infty) > 0$ for any $x \in E$ and finely open G , weakly irreducible if $\mathbf{P}^x(T_G < \infty) > 0$ for any $x \in E$ and open G .*

The weak irreducibility is weaker than the irreducibility.

Example 1. Let ν be a probability measure charging on all non-zero rationals and X the compound Poisson process with Lévy measure ν . Then any point is finely open. Since rational numbers are dense, X is weakly irreducible, but not irreducible with reason that X , starting from 0, can only reach rational numbers.

Theorem 2.1. *The following are equivalent*

1. *X is irreducible;*
2. *For any $A \in \mathcal{E}$, $U^\alpha 1_A$ is either 0 identically or positive everywhere;*
3. *All non-trivial excessive measures are equivalent.*

Proof. We may assume $\alpha = 0$. Suppose (1) is true. If $U1_A$ is not identically zero, then there exists $\delta > 0$ such that $D := \{U1_A > \delta\}$ is non-empty. Since $U1_A$ is excessive and thus finely continuous, D is finely open and the fine closure of D is contained in $\{U1_A \geq \delta\}$. Then by Proposition II.2.8 and Theorem I.11.4[1],

$$U1_A(x) \geq P_D U1_A(x) = \mathbf{E}^x(U1_A(X_{T_D})) \geq \delta \mathbf{P}^x(T_D < \infty) > 0.$$

Conversely suppose (2) is true. Then for any finely open set D , by the right continuity of X , $U1_D(x) > 0$ for any $x \in D$. Therefore $U1_D$ is positive everywhere on E .

Let ξ be a non-trivial excessive measure. Since $\alpha \xi U^\alpha \leq \xi$, $\xi(A) = 0$ implies that $\xi U^\alpha(A) = 0$. However ξ is non-trivial. Thus it follows from (2) that $U^\alpha 1_A \equiv 0$, i.e., A is potential zero. Conversely if A is potential zero, then $\xi(A) = 0$ for any excessive measure ξ . Therefore (2) and (3) are equivalent. \square

It is well-known that the strong Feller property and (weak) irreducibility together imply the uniqueness of invariant distribution, which implies the ergodicity. By the strong Feller, we mean that P_t takes bounded measurable functions to continuous

functions. A condition obviously weaker than strong Feller is called LSC, which means that for any measurable set B , $U^\alpha(\cdot, B)$ is lower semi-continuous. The Brownian motion is strong Feller, but compound Poisson process is not strong Feller.

Lemma 2.1. *If X satisfies LSC and weak irreducibility, then it is irreducible.*

Proof. Let $A \in \mathcal{E}$. $U^\alpha 1_A \neq 0$ identically. There is $b > 0$ such that $G = \{U^\alpha 1_A > b\} \neq \emptyset$ and is open due to the property LSC. Since $U^\alpha 1_A$ is α -excessive, we have by Proposition II.2.8 [1] for any $x \in E$,

$$U^\alpha 1_A(x) \geq P_G^\alpha U^\alpha 1_A(x) = \mathbf{P}^x(e^{-\alpha T_G} \cdot U^\alpha 1_A(X(T_G))).$$

But $X(T_G) \in \bar{G}$ by Theorem I.11.4[1] and then $X(T_G) \geq b$. Hence by the weak irreducibility, we have

$$U^\alpha 1_A(x) \geq b \mathbf{E}^x(e^{-\alpha T_G}, T_G < \infty) > 0.$$

□

Example 2. Let $N = (N_t)$ be a Poisson process with parameter $\lambda > 0$ and $X_t = N_t - t$. Then X does not satisfy strong Feller but satisfies LSC. Hence it is irreducible. Since X jumps forward and drifts backward, any set such as $(a, b]$ is finely open. It may be shown that it satisfies a stronger irreducibility $\mathbf{P}^x(T_y < \infty) > 0$ for any $x, y \in \mathbb{R}$.

The strong Feller property is certainly too much and hence is not essential for uniqueness of stationary distribution as indicated in the following very simple example.

Example 3. Consider the uniform translation X on unit circle. Then X is not strong Fellerian. However the uniform distribution on circle is the only stationary distribution of X .

It seems that in studying uniqueness problems, our irreducibility is more natural than the irreducibility under the original topology. In the joint paper [4] of the second author, it is proven that our irreducibility implies the uniqueness of (σ -finite) symmetrizing measures for Markov processes. Here we shall prove the uniqueness of invariant distribution under irreducibility.

Theorem 2.2. *The irreducibility of X implies the uniqueness of invariant distribution.*

Proof. We complete the proof in 4 steps.

1. ([3], Theorem 3.2.4) If a probability measure μ is invariant, then μ is ergodic if and only if for any $A \in \mathcal{E}$ and $t > 0$, $P_t 1_A = 1_A$, μ -a.s. implies that $\mu(A) = 0$ or 1.

2. ([3], Theorem 3.2.5) If μ and ν are ergodic to (P_t) and $\mu \neq \nu$, then μ and ν are singular.

Note that the condition for 1 and 2 is the stochastic continuity of X which is surely implied by the right continuity of X .

3. If (P_t) is irreducible, then any invariant distribution is ergodic.

In fact for any $A \in \mathcal{E}$, $t > 0$,

$$P_t 1_A = 1_A, \mu - \text{a.s.}$$

Then for any $B \in \mathcal{E}$,

$$\langle 1_B, P_t 1_A \rangle_\mu = \langle 1_B, 1_A \rangle_\mu$$

for any $t > 0$. It follows by Fubini's theorem that

$$\langle 1_B, \alpha U^\alpha 1_A \rangle_\mu = \langle 1_B, 1_A \rangle_\mu$$

for any $\alpha > 0$. Hence $\alpha U^\alpha 1_A = 1_A$, μ -a.s. Our irreducibility is equivalent to $U^\alpha 1_A$ is either 0 identically or positive everywhere. This implies that $\mu(A) = 0$ or 1.

4. Irreducibility implies the uniqueness of invariant distribution.

In fact, if μ and ν are two different invariant distributions, then they are ergodic by (3) and they are singular by (2). However the irreducibility implies that all excessive measures are equivalent to each other, which leads to a contradiction.

That completes the proof. □

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